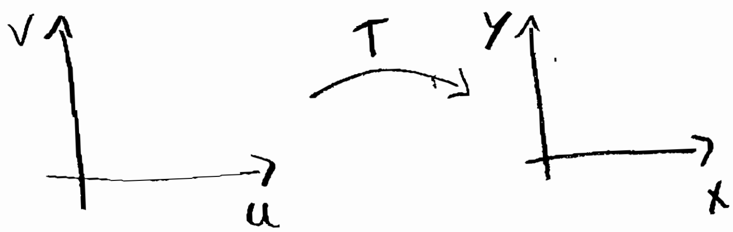
"Change of Variables" in $\mathbb{R}^2$ [Andrew Critch, Math 53 09Su]

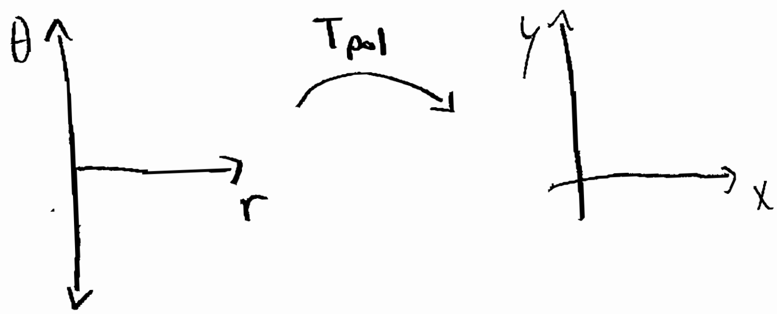
(Works exactly the same in $\mathbb{R}^n$!)

A "change of variables" in $\mathbb{R}^2$, or "u,v substitution" (any letters are fine!) is a map $T : \boxed{(\mathbb{R}^2) \to \mathbb{R}^2}$ "part of $\mathbb{R}^2$"

where we think of the first $\mathbb{R}^2$ as a "u,v" plane and the second as an "x,y" plane — $T : \boxed{(\mathbb{R}^2_{u,v}) \to \mathbb{R}^2_{x,y}}$



Example: polar coordinates!



$T_{pol}(r,\theta) = \boxed{\begin{bmatrix} r\cos\theta \\ r\sin\theta \end{bmatrix}} = \begin{bmatrix} x \\ y \end{bmatrix}$, $dy\,dx = r\,dr\,d\theta$ "stretch factor"

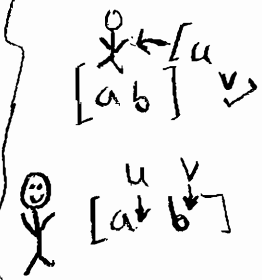
Let's study linear changes of variables first!

Linear C.O.V. and matrices

Recall that $[a\ b] \overset{\text{no dot!}}{} \begin{bmatrix} u \\ v \end{bmatrix} \overset{\text{means}}{=\!=} \boxed{au + bv}$

(and $\begin{bmatrix} a \\ b \end{bmatrix} \cdot$ means $\boxed{[a\ b]}$), and we can think of $[a\ b]$ as a linear function $\mathbb{R}^2 \to \mathbb{R}^1$ taking $\boxed{\begin{bmatrix} u \\ v \end{bmatrix}}$ as an input in this way.

Imagine pulling "$\begin{bmatrix} u \\ v \end{bmatrix}$" up over the top of "$[a\ b]$":



$$\text{Similarly, } \underbrace{\begin{bmatrix} a & b \\ c & d \end{bmatrix}}_{\text{a "matrix", } M} \underbrace{\begin{bmatrix} u \\ v \end{bmatrix}}_{\text{input}} \overset{\text{means}}{=\!=} \underbrace{\begin{bmatrix} au + bv \\ cu + dv \end{bmatrix}}_{\text{output}} \in \mathbb{R}^2$$

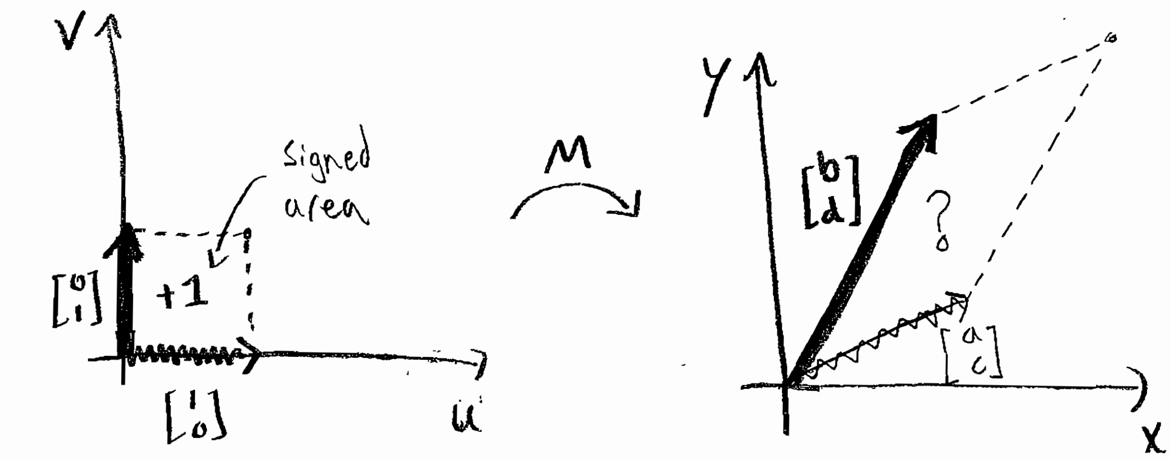
one number! Don't get confused because it takes up a lot of space!

$$\text{So, } M = \begin{bmatrix} a & b \\ c & d \end{bmatrix} : \mathbb{R}^2 \xrightarrow{\text{linear}} \mathbb{R}^2$$

(this is almost everything matrices are for!)

Question: how much does M "stretch" things? does it "flip" things?

Answer: Consider the (ordered) unit square, $\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix}$:

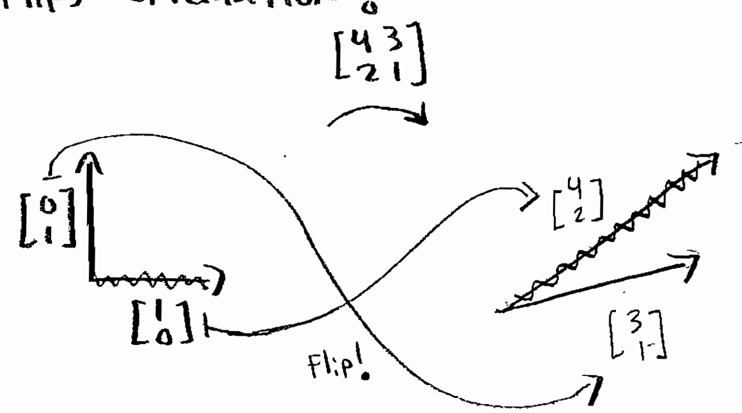


(check: $M\begin{bmatrix}1\\0\end{bmatrix} = \begin{bmatrix}a & b\\c & d\end{bmatrix}\begin{bmatrix}1\\0\end{bmatrix} = \boxed{\begin{bmatrix}a\\c\end{bmatrix}}$ and $M\begin{bmatrix}0\\1\end{bmatrix} = \boxed{\begin{bmatrix}b\\d\end{bmatrix}}$)

We already learned that the signed area of the picture at the right is $\boxed{\det\begin{pmatrix}a & b\\c & d\end{pmatrix} = \det M}$!

So $\det M$ is a "signed stretch factor" for M.

Example: $M = \begin{bmatrix}4 & 3\\2 & 1\end{bmatrix}$ has $\det M = \boxed{4-6=2}$. It stretches by a factor of $\boxed{2}$ and the "$-$" sign tells you that it "flips" orientation:



Non-Linear C.O.V.: Stretch factor comes from a Linear Approximation Formula!

Say $T(u,v) = \begin{bmatrix} x(u,v) \\ y(u,v) \end{bmatrix}$ is any change of variables in $\mathbb{R}^2$

(For example, $x = u^2 + v$, $y = uv^2$)

The Jacobian (function) matrix of T is defined by

$$\underbrace{\mathcal{J}}_{\text{fancy "J"}}{}_T = \begin{bmatrix} x_u & x_v \\ y_u & y_v \end{bmatrix}, \quad \text{Functions of } (u,v)$$

and the Jacobian (function) determinant is its determinant:

$$\underbrace{J}_{\text{regular "J"}}{}_T = \boxed{x_u y_v - x_v y_u} = \text{"}\frac{\partial(x,y)}{\partial(u,v)}\text{" in text.}$$

Why do we do this? Because the Jacobian matrix gives us a Linear Approximation Formula for T!

Thm: $$T(\vec{u}) - T(\vec{u}_0) \overset{\vec{u}\to\vec{u}_0}{\sim} \overbrace{\mathcal{J}_T(\vec{u}_0)}^{\text{Constant Matrix}} (\vec{u} - \vec{u}_0) \quad \text{no dot!}$$

$$\vec{u} = \begin{bmatrix} u \\ v \end{bmatrix}, \quad \vec{u}_0 = \begin{bmatrix} u_0 \\ v_0 \end{bmatrix} = \begin{bmatrix} a \\ b \end{bmatrix} \text{ in text}$$

Recall: LHS $\overset{\vec{u}\to\vec{u}_0}{\sim}$ RHS means

$$\boxed{\lim_{\vec{u}\to\vec{u}_0} \left(\frac{\text{LHS} - \text{RHS}}{\|\vec{u} - \vec{u}_0\|} \right) = 0}$$

Example Say $T(u,v) = \begin{bmatrix} u^2+v \\ uv^2 \end{bmatrix}$. Write the Jacobian

LAF for T at the input $\begin{bmatrix} u_0 \\ v_0 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$:

(1) $T(\vec{u}_0) = T(1,2) = \boxed{\begin{bmatrix} 3 \\ 4 \end{bmatrix}}$

(2) $\mathcal{J}_T = \begin{bmatrix} x_u & x_v \\ y_u & y_v \end{bmatrix} = \begin{bmatrix} 2u & 1 \\ v^2 & 2uv \end{bmatrix}$

nothing to plug in!

(3) $\mathcal{J}_T(\vec{u}_0) = \mathcal{J}_T(1,2) = \begin{bmatrix} 2 & 1 \\ 4 & 4 \end{bmatrix}$

LAF: $\begin{bmatrix} u^2+v \\ uv^2 \end{bmatrix} - \boxed{\begin{bmatrix} 3 \\ 4 \end{bmatrix}} \overset{\boxed{\substack{u \to 1 \\ v \to 2}}}{\sim} \boxed{\begin{bmatrix} 2 & 1 \\ 4 & 4 \end{bmatrix} \begin{bmatrix} u-1 \\ v-2 \end{bmatrix}} = \boxed{\begin{bmatrix} 2u+v-4 \\ 4u+4v-12 \end{bmatrix}}$

$\therefore \begin{bmatrix} u^2+v \\ uv^2 \end{bmatrix} \overset{\boxed{\substack{u \to 1 \\ v \to 2}}}{\sim} \boxed{\begin{bmatrix} 2u+v-1 \\ 4u+4v-8 \end{bmatrix}}$

Remark: $\nabla(u^2+v) = [2u \;\; 1]$, so it's like we used the old LAF on x and y separately!

Here, $\boxed{\begin{bmatrix} -1 \\ -8 \end{bmatrix}}$ is a translation, and

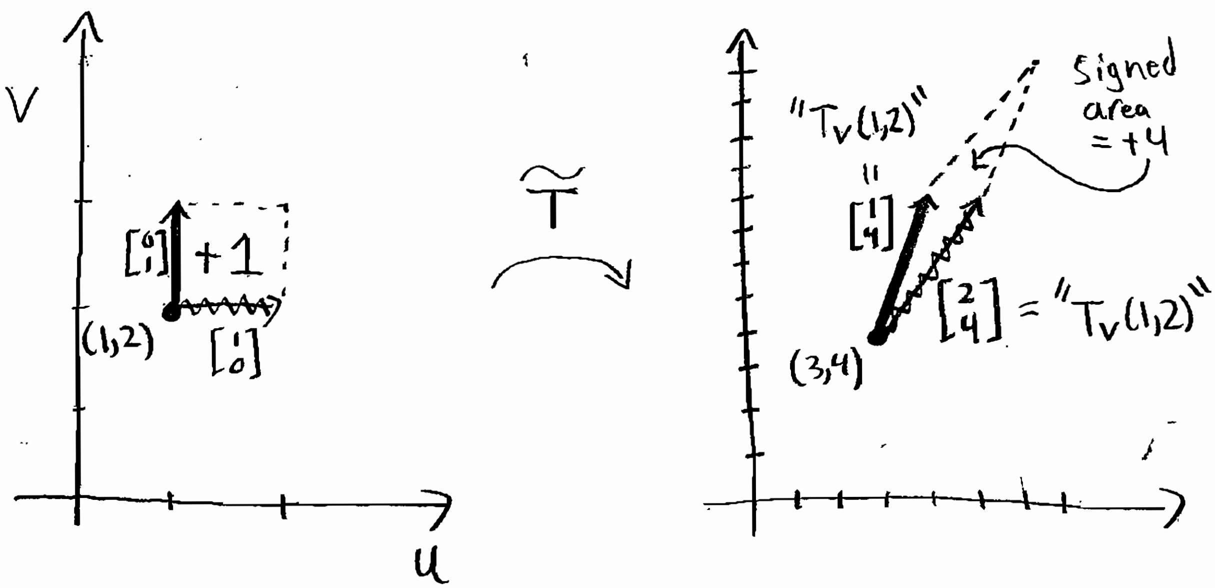
$J_T(1,2) = \boxed{\det\begin{pmatrix} 2 & 1 \\ 4 & 4 \end{pmatrix}} = \boxed{+4}$

is the signed stretch factor

for T near the input $(u_0, v_0) = (1,2)$!

(See next page for a diagram summary of these relationships.)

Example cont'd | Let's denote the (translated) linear approximation $\begin{bmatrix} 2u+v-1 \\ 4u+4v-8 \end{bmatrix}$ by $\tilde{T}(u,v)$. The following picture summarizes the relationship between $\vec{u}_0 = (1,2)$, $T(\vec{u}_0) = (3,4)$, $J_T(1,2) = \begin{bmatrix} 2 & 1 \\ 4 & 4 \end{bmatrix}$, and $J_T(1,2) = +4$ via the effect of $\tilde{T}$ on an ordered unit square starting at $\vec{u}_0$:



Notice again the columns of the matrix $J_T(1,2) = \begin{bmatrix} 1 & 2 \\ 4 & 4 \end{bmatrix}$ occurring as parallelogram sides. The columns $\begin{bmatrix} x_u \\ y_u \end{bmatrix}$ and $\begin{bmatrix} x_v \\ y_v \end{bmatrix}$ are sometimes denoted "T_u" and "T_v" (more of this in section 16.6!).

Practice with Jacobians (submit with homework 15.9)*

In each problem, find a) J_T, b) the (trans.) linear approximation for T at the given input, c) $|J_T|$ the function d) $|J_T|$(the given input)

#1. $T(u,v) = \begin{bmatrix} 5u-v \\ u+3v \end{bmatrix}$, $\vec{u}_0 = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$

#2. $x = uv$, $y = \frac{u}{v}$, $\vec{u}_0 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

#3. $T(r,\theta) = \begin{bmatrix} e^{-r}\cos\theta \\ e^{r}\sin\theta \end{bmatrix}$, $\begin{bmatrix} r_0 \\ \theta_0 \end{bmatrix} = \begin{bmatrix} 1 \\ \pi \end{bmatrix}$

#4. $x = e^{s+t}$, $y = e^{s-t}$, $\begin{bmatrix} s_0 \\ t_0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

#5. $T(u,v,w) = \begin{bmatrix} u/v \\ v/w \\ w/u \end{bmatrix}$, $\vec{u}_0 = \begin{bmatrix} u_0 \\ v_0 \\ w_0 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix}$

#6. $x = v+w^2$, $y = w+u^2$, $z = u+v^2$, $\vec{u}_0 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

in $\mathbb{R}^3$! use 3x3 matrices!

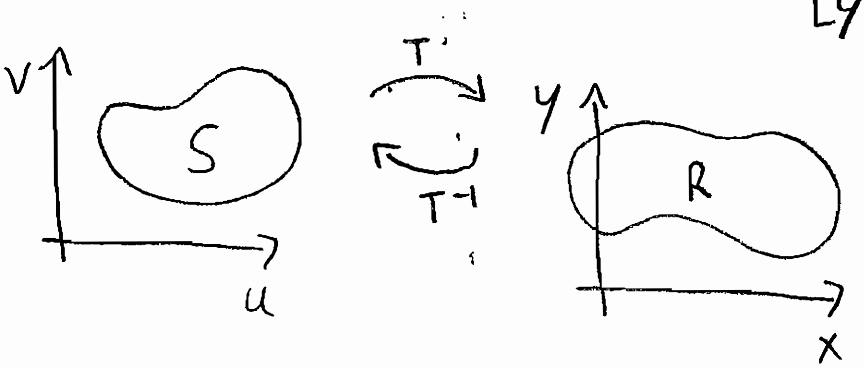
* These problems are extended versions of #1-6 in 15.9, so just replace them with these in your homework :)

Change of Variables in Integrals

Ⓜ Stewart 15.9 summary:

1) You have an integral $\iint_R f \, dy\,dx$ where $f = f(x,y)$

2) You have, or choose, a C.O.V. $T(u,v) = \begin{bmatrix} x(u,v) \\ y(u,v) \end{bmatrix}$



Need $T: S \to R$ to be $\boxed{\text{one-to-one}}$ ("injective")
and $\boxed{\text{onto}}$ ("surjective")
<u>and</u> C^1 (continuous first partials)

3) Hopefully can solve $x = x(u,v)$, $y = y(u,v)$ for (u,v) to get $\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} u(x,y) \\ v(x,y) \end{bmatrix} = T^{-1}(x,y)$.

4) Determine region "S" by subbing u,v into def'n of R (start with boundary equations of R).

Thm:

$$\iint\limits_R \overset{\boxed{f(x,y)}}{f}\,dy\,dx = \iint\limits_S \overset{\boxed{f(x(u,v),y(u,v))}}{f}\,|J_T|\,du\,dv$$

absolute value!

$\boxed{\det \mathbb{J}_T}$

Remarks: ☆ If the COV is decent, then J_T is hopefully always positive or always negative, so $|J_T| = J_T$ or $-J_T$ for all u, v.

☆ The stretch factors $\boxed{\text{"}r\text{", "}r\text{" and "}\rho^2\sin\varphi\text{"}}$ from polar, cylindrical and spherical coordinates are all Jacobian determinants! (see Example 4, p. 1019)

☆ Just like pol/cyl/sph coords, a change of variables can be made to make either 1) The $\boxed{\text{Function}}$ nicer or 2) the $\boxed{\text{region}}$ nicer, and ideally, both!

☆ The only way to get good at this is to

$\boxed{\text{PRACTICE!!}}$:)

So, hop to it!! :)